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On the uncontrollable damped triple inverted pendulum

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Abstract

In this paper, the controllability of the damped triple inverted pendulum is investigated. The work is concerned with the form of the cancelling pole and zero which appear in the transfer functions of an uncontrollable system, and follows on from earlier work on the damped double inverted pendulum. The investigation considers first the cases where only one of the three arm frictions is non-zero, and then explores the cases when two of the three arm frictions are non-zero. Due to the complexity of this problem, and the difficulties with the symbolic manipulation software, exploratory numerical investigations have been carried out to facilitate the symbolic investigations, all of which are reported here.

Keywords: Pendulum; Controllability; Symbolic computation

1. Introduction

The balancing of an inverted pendulum is a classic control problem of some 30 years or so standing, which has been revisited of late in the light of recent developments in symbolic computation. The controllability of the damped double inverted pendulum has previously been investigated with the aid of algebraic computation [2,3,6], and some interesting results have been obtained. In particular, the transfer functions with respect to the possible control inputs, for three balancing states, have been calculated symbolically, and the cancelling poles have been extracted in full algebraic form. In this paper we address the more complex problem of the triple-inverted pendulum, which has, not a curve of uncontrollability (with respect to the force on the trolley), but a surface of uncontrollability.

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The increased complexity of this problem has necessitated an exploratory numerical investigation of the transfer functions. The numerical results for the cancelling pole, along with some deductions concerning its symbolic form, have facilitated a symbolic investigation.

2. Pendulum model

2.1. The system

Consider the system shown in Fig. 1. Three links, $l^{(1)}, l^{(2)}, l^{(3)}$, of respective mass and length m_1, l_1 (kg, m), m_2, l_2 and m_3, l_3, are supported by a trolley of mass M (kg), and the whole pendulum operates, in the plane, under the action of a single controlling input along the line of $\hat{\imath}$—the given force of magnitude $|\pm F(t)|$—together with gravity, which acts downward and is denoted by g (m s^{-2}). It is further assumed that (nonzero) linear friction forces oppose any system motion along the horizontal track and at the base of each link; the respective coefficients of damping under this assumption are written as λ_x (kg s^{-1}) and $\lambda_1, \lambda_2, \lambda_3$ (kg m^2 s^{-1}), each assumed < 0 as part of the overall consideration of the dynamics of the problem.

With reference to Fig. 1, the coordinates employed are $x(t), \theta_1(t), \theta_2(t), \theta_3(t)$, where x gives the distance of the base of the pendulum from an arbitrary point on the track x_0, say, and $\theta_1, \theta_2, \theta_3$ give a measure of the angular displacement of each link from a local vertical, taken positive clockwise. The state vector $x(t)$ comprises the four coordinates and their time derivatives

$$\mathbf{x}(t) = [x(t) \quad \theta_1(t) \quad \theta_2(t) \quad \theta_3(t) \quad \dot{x}(t) \quad \dot{\theta}_1(t) \quad \dot{\theta}_2(t) \quad \dot{\theta}_3(t)]^{\mathrm{T}}. \tag{1}$$

2.2. The balancing problems

Denote any system state of unstable equilibrium, at some arbitrary point along the track, by $S_{\theta_1,\theta_2,\theta_3}$, where $\theta_1, \theta_2, \theta_3$ define the respective alignment of each link $l^{(1)}, l^{(2)}, l^{(3)}$ as stated. There are three such states for the triple-inverted pendulum, written $S_{0,0,0}, S_{0,\pi,0}, S_{0,0,\pi}$, each of which, in the context of control, sets a viable balancing problem [1]. The first of these states, $S_{0,0,0}$, provides the standard one which appears in the overwhelming majority of work, whilst the other two may be regarded as nonstandard. We refer to these states as, in order, "up–up–up", "up–down–up" and "up–up–down", and henceforth the respective control assignments attached to them Balancing Problems I, II, III.

3. Controllability theory

3.1. Kalman controllability

Formulation of full-nonlinear equations of motion of the system is straightforward, and the system is linearised about a state of unstable equilibrium in the usual way. The well-known Kalman controllability test gives the theoretical feasibility of control with respect to an appropriate control input. Each balancing problem gives rise to a linear system

$$\frac{\mathrm{d}(\delta\mathbf{x})}{\mathrm{d}t} = [A]\delta x + \underline{\mathbf{b}}\,\delta F, \tag{2}$$

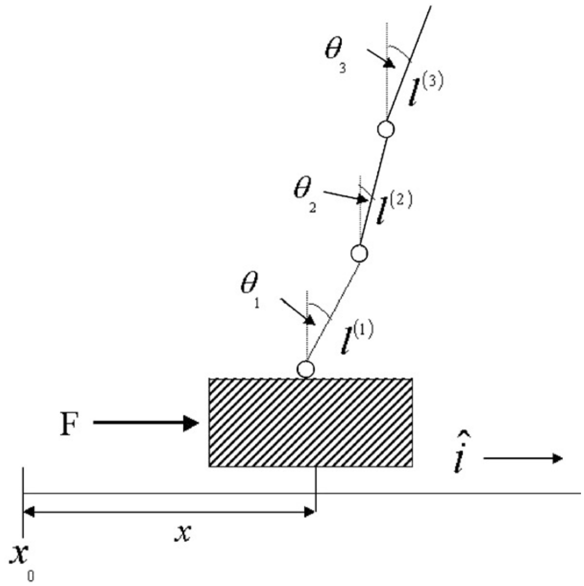

Fig. 1. The damped translating triple pendulum.

where

$$A = \begin{bmatrix} \mathbf{0}_{4,4} & \vdots & \mathbf{I}_4 \\ \cdots & \cdots & \cdots \\ \mathbf{A}^{\dagger} & \vdots & \mathbf{A}^{*} \end{bmatrix}, \quad \underline{\mathbf{b}} = \begin{bmatrix} \mathbf{0}_{4,1} \\ \cdots \\ \mathbf{b}^{\dagger} \end{bmatrix} \tag{3}$$

for perturbations $\delta x, \delta F$ in x, F.

Now, construction of the appropriate (8×8) controllability matrix

$$C = [\underline{\mathbf{b}} \vdots [A]\underline{\mathbf{b}} \vdots [A^2]\underline{\mathbf{b}} \vdots [A^3]\underline{\mathbf{b}} \vdots [A^4]\underline{\mathbf{b}} \vdots [A^5]\underline{\mathbf{b}} \vdots [A^6]\underline{\mathbf{b}} \vdots [A^7]\underline{\mathbf{b}}] \tag{4}$$

pertaining to F yields, by evaluation of its determinant, a necessary and sufficient condition for pendulum controllability. This is found to involve a function of general form

$$c = c(\lambda_1, \lambda_2, \lambda_3; m_1, m_2, m_3, l_1, l_2, l_3, g), \tag{5}$$

describing a surface above the $(\lambda_1, \lambda_2, \lambda_3)$ space, the 'friction space', for any particular system. It is worth noting that both λ_x and M have been observed to be redundant controllability variables in this problem.

3.2. Transfer functions

The system equations may be restated in transfer function form, by taking the Laplace Transform of Eq. (2).

The (linearised) output equation

$$\delta Y = [D]\delta X \tag{6}$$

accompanying (2) will be assumed, where

$$D = [\mathbf{I}_4 \;\vdots\; \mathbf{0}_{4,4}], \tag{7}$$

so that the outputs of the system are simply the respective states x, θ_1, θ_2, and θ_3. From standard theory, uncontrollability manifests itself as pole-zero cancellation in the transfer functions, and it is this phenomena which will be investigated in this paper.

The balancing problem being examined is that set by the demanded state $S_{0,0,0}$, the standard one common in control. For simplicity, the case when $M = m_1 = m_2 = m_3 = \varepsilon_m$ and $l_1 = l_2 = l_3 = \varepsilon_l$, the quasi-uniform system, will be considered. The nontrivial partitioned blocks of A and $\underline{\mathbf{b}}$ are found, by computation, to have symbolic entries:

$$\begin{aligned}
&A^{\dagger}_{11} = 0, && A^{\dagger}_{21} = 0,\\
&A^{\dagger}_{12} = -165g/67, && A^{\dagger}_{22} = 480g/(67\varepsilon_l),\\
&A^{\dagger}_{13} = 27g/67, && A^{\dagger}_{23} = -243g/(67\varepsilon_l),\\
&A^{\dagger}_{14} = -3g/67, && A^{\dagger}_{24} = 27g/(67\varepsilon_l),\\
&A^{\dagger}_{31} = 0, && A^{\dagger}_{41} = 0,\\
&A^{\dagger}_{32} = -405g/(67\varepsilon_l), && A^{\dagger}_{42} = 135g/(67\varepsilon_l),\\
&A^{\dagger}_{33} = 450g/(67\varepsilon_l), && A^{\dagger}_{43} = -351g/(67\varepsilon_l),\\
&A^{\dagger}_{34} = -117g/(67\varepsilon_l), && A^{\dagger}_{44} = 240g/(67\varepsilon_l),\\
&A^{*}_{11} = 52\lambda_x/(67\varepsilon_m), && A^{*}_{21} = -66\lambda_x/(67\varepsilon_m\varepsilon_l),\\
&A^{*}_{12} = -6(11\lambda_1 + 14\lambda_2)/(67\varepsilon_m\varepsilon_l), && A^{*}_{22} = 6(32\lambda_1 + 59\lambda_2)/(67\varepsilon_m\varepsilon_l^2),\\
&A^{*}_{13} = 12(7\lambda_2 + 2\lambda_3)/(67\varepsilon_m\varepsilon_l), && A^{*}_{23} = -6(59\lambda_2 + 36\lambda_3)/(67\varepsilon_m\varepsilon_l^2),\\
&A^{*}_{14} = -24\lambda_3/(67\varepsilon_m\varepsilon_l), && A^{*}_{24} = 216\lambda_3/(67\varepsilon_m\varepsilon_l^2),\\
&A^{*}_{31} = 18\lambda_x/(67\varepsilon_m\varepsilon_l), && A^{*}_{41} = -6\lambda_x/(67\varepsilon_m\varepsilon_l),\\
&A^{*}_{32} = -6(27\lambda_1 + 77\lambda_2)/(67\varepsilon_m\varepsilon_l^2), && A^{*}_{42} = 18(3\lambda_1 + 16\lambda_2)/(67\varepsilon_m\varepsilon_l^2),\\
&A^{*}_{33} = 6(77\lambda_2 + 89\lambda_3)/(67\varepsilon_m\varepsilon_l^2), && A^{*}_{43} = -6(48\lambda_2 + 119\lambda_3)/(67\varepsilon_m\varepsilon_l^2),\\
&A^{*}_{34} = -534\lambda_3/(67\varepsilon_m\varepsilon_l^2), && A^{*}_{44} = 714\lambda_3/(67\varepsilon_m\varepsilon_l^2),
\end{aligned}$$

and

$$\begin{aligned}
&b^{\dagger}_1 = 52/(67\varepsilon_m),\\
&b^{\dagger}_2 = -66/(67\varepsilon_m\varepsilon_l),\\
&b^{\dagger}_3 = 18/(67\varepsilon_m\varepsilon_l),\\
&b^{\dagger}_4 = -6/(67\varepsilon_m\varepsilon_l).
\end{aligned}$$

4. Numerical results ($\lambda_i \neq 0$)

The uncontrollability condition, which is determined by evaluating the determinant of C in Eq. (4), is a highly nonlinear equation. In the first instance, we will consider the three cases where only one of the three frictions λ_1, λ_2 or λ_3 is nonzero [4].

4.1. $\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 \neq 0$

In this case the determinant of C is given by

$$\det(\mathbf{C}) = 460\lambda_3^2 + 529g\varepsilon_m^2\varepsilon_l^2.$$

Clearly, since ε_m, ε_l and λ_3 are all real quantities, the determinant can never be zero, and so the system will always be controllable, and there will be no pole-zero cancellation in the transfer functions.

4.2. $\lambda_1 = 0$ $\lambda_2 \neq 0$ $\lambda_3 = 0$

In this case the determinant of C is given by

$$\det(\mathbf{C}) = 18711\lambda_2^2 - 2116g\varepsilon_m^2\varepsilon_l^3,$$

so for uncontrollability,

$$\lambda_2 = -\sqrt{\frac{2116g\varepsilon_m^2\varepsilon_l^3}{18711}} = -\frac{46\varepsilon_m\varepsilon_l}{2079}\sqrt{231g\varepsilon_l}. \tag{8}$$

Obtaining the transfer functions symbolically gives the set of ratios of polynomials, which must contain a common factor.

However, the factorisation process available in the symbolic manipulation package has difficulties with the fractional powers which occur throughout the transfer functions, and is therefore unable to extract the cancelling pole. Thus, a numerical investigation of the cancelling pole has been undertaken to enable the form of the cancelling pole to be deduced. For simplicity, at this stage, g, the acceleration due to gravity, is taken to be 10 m/s^2:

ε_l	ε_m	Cancelling pole
0.50	0.50	$7s - 68 \;= 7s - 2\sqrt{231}\sqrt{5}$
0.50	1.00	$7s - 68 \;= 7s - 2\sqrt{231}\sqrt{5}$
1.00	0.50	$7s - 48 \;= 7s - \sqrt{231}\sqrt{10}$
1.00	2.00	$7s - 48 \;= 7s - \sqrt{231}\sqrt{10}$
0.10	1.00	$7s - 152 = 7s - 10\sqrt{231}\sqrt{5}$

It has been deduced that the cancelling pole may be written as

$$7s - \frac{1}{\varepsilon_l}\sqrt{231}\sqrt{g\varepsilon_l}. \tag{9}$$

4.3. $\lambda_1 \neq 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

In this case the determinant of C is given by

$$\det(\mathbf{C}) = 297\lambda_1^4 - 1974g\varepsilon_m^2\varepsilon_l^3\lambda_1^2 + 2645g^2\varepsilon_m^4\varepsilon_l^6,$$

so for uncontrollability, since λ_1 is negative,

$$\lambda_1 = -\sqrt{\frac{g\varepsilon_m^2\varepsilon_l^3(987 \pm 78\sqrt{31})}{297}}.$$

Since either choice for λ_1 is valid, we elect to use

$$\lambda_1 = -\sqrt{\frac{g\varepsilon_m^2\varepsilon_l^3(987 - 78\sqrt{31})}{297}}. \tag{10}$$

Once again, the factorisation process available in the symbolic manipulation package has difficulties with the fractional powers which occur throughout the transfer functions, and is unable to extract the cancelling pole. Again, a numerical investigation of the cancelling pole has been undertaken, and the form of the cancelling pole has therefore been able to be deduced. Once again, g, the acceleration due to gravity, is taken to be 10 m/s^2, and, since it does not affect the form of the cancelling pole, ε_m has been taken to be 1:

ε_l	Cancelling pole
0.10	$253s - (100 + 10\sqrt{31})\sqrt{3619 - 286\sqrt{31}}\sqrt{1}$
0.20	$253s - (50 + 5\sqrt{31})\sqrt{3619 - 286\sqrt{31}}\sqrt{2}$
0.5	$253s - (20 + 2\sqrt{31})\sqrt{3619 - 286\sqrt{31}}\sqrt{5}$
1.0	$253s - (10 + \sqrt{31})\sqrt{3619 - 286\sqrt{31}}\sqrt{10}$

It has been deduced that the cancelling pole may be written as

$$s - \frac{(10 + \sqrt{31})}{253\varepsilon_l}\sqrt{3619 - 286\sqrt{31}}\sqrt{g\varepsilon_l}. \tag{11}$$

Using Eq. (10), and substituting for

$$\sqrt{329 - 26\sqrt{31}}\sqrt{g\varepsilon_l}$$

in Eq. (11) gives

$$s + \frac{3(10 + \sqrt{31})}{23}\frac{\lambda_1}{\varepsilon_m\varepsilon_l^2}. \tag{12}$$

5. Symbolic results

Having obtained, numerically, the form of the cancelling pole for the two conditionally uncontrollable cases, it is now appropriate to revisit the symbolic forms of the transfer functions. Since it

```
The 1 numerator is:
1/317388946016*(297*r2+46*s*em*el^2)*
(5355037376*el^10*em^5*s^5-88949140896*
r2*el^8*em^4*s^4-97740625752*r2^2*el^6*
em^3*s^3+2466888402132*r2^3*el^4*em^2*s^2+
305988729354*el^2*s*r2^4*em-8932840308315*
r2^5)/em^7/el^12

The 2 numerator is:
-3/149994776*s^2*(297*r2+46*s*em*el^2)*
(1070696*s^3*em^3*el^6-12753132*r2*el^4*
em^2*s^2-5451138*el^2*em*r2^2*s+159137055*
r2^3)/em^5/el^9

The 3 numerator is:
9/149994776*s^2*(297*r2+46*s*em*el^2)*
(97336*s^3*em^3*el^6+1318268*r2*el^4*
em^2*s^2+1817046*el^2*em*r2^2*s-53045685*
r2^3)/em^5/el^9

The 4 numerator is:
-3/149994776*s^2*(297*r2+46*s*em*el^2)*
(97336*s^3*em^3*el^6+4043676*r2*el^4*em^2*
s^2-5451138*el^2*em*r2^2*s+159137055
*r2^3)/em^5/el^9

The denominator is:
1/158694473008*s^2*(297*r2+46*s*em*el^2)*
(3449879848*el^10*em^5*s^5-64290671340*r2*
el^8*em^4*s^4-117622647450*r2^2*el^6*em^3*
s^3+2660822247699*r2^3*el^4*em^2*s^2+
611977458708*el^2*s*r2^4*em-17865680616630*
r2^5)/em^6/el^12
```

Fig. 2. Symbolic results for $\lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 = 0$.

is apparently the fractional powers which cause the manipulation package problems, the uncontrollability conditions need to be rearranged to avoid their use.

5.1. $\lambda_1 = 0\ \lambda_2 \neq 0\ \lambda_3 = 0$

In this case, Eq. (8) with λ_2 as the subject clearly contains fractional powers of g and ε_l. If we substitute for ε_l or ε_m fractional powers will also be present, so consider substituting for g:

$$g = \frac{18711\lambda_2^2}{2116\varepsilon_m^2\varepsilon_l^3}.$$

After obtaining the transfer functions and factorising, we obtain, from Matlab, the results shown in Fig. 2. The common cancelling pole is

$$46s\varepsilon_m\varepsilon_l^2 + 297\lambda_2.$$

Substituting for $\sqrt{g\varepsilon_l}$ from Eq. (8) into Eq. (9), and rearranging gives

$$7s + \frac{2079\lambda_2}{46\varepsilon_m\varepsilon_l^2} = \frac{7}{46\varepsilon_m\varepsilon_l^2}(46s\varepsilon_m\varepsilon_l^2 + 297\lambda_2), \tag{13}$$

so the form of the pole deduced from the numerical results is correct.

The x numerator contains the cancelling factor into a fifth-order polynomial in s, whilst the θ_1, θ_2 and θ_3 numerators are all in the form

$$s^2(46s\varepsilon_m\varepsilon_l^2 + 297\lambda_2)p(s^3)$$

and are clearly closely related to each other. The denominator has s^2 and the cancelling factor into a fifth-order polynomial.

5.2. $\lambda_1 \neq 0$ $\lambda_2 = 0$ $\lambda_3 = 0$

In this case, Eq. (10) with λ_1 as the subject clearly contains fractional powers of g and ε_l. If we substitute for ε_l or ε_m fractional powers will also be present, so consider substituting for g:

$$g = \frac{297\lambda_1^2}{(987 - 78\sqrt{31})\varepsilon_m^2\varepsilon_l^3}.$$

After obtaining the transfer functions and factorising, we obtain, from Matlab, the results shown in Fig. 3. The common cancelling pole is

$$23s\varepsilon_m\varepsilon_l^2 + (30 + 3\sqrt{31})\lambda_1.$$

Rearranging Eq. (12) gives

$$\frac{1}{23\varepsilon_m\varepsilon_l^2}(23s\varepsilon_m\varepsilon_l^2 + 3(10 + \sqrt{31})\lambda_1), \tag{14}$$

so the form of the pole deduced from the numerical results is correct.

The x numerator contains the cancelling factor into a fifth-order polynomial in s, whilst the θ_1 numerator is of the form:

$$(23s\varepsilon_m\varepsilon_l^2 - 3(10 + \sqrt{31})\lambda_1)(23s\varepsilon_m\varepsilon_l^2 + 3(10 + \sqrt{31})\lambda_1)p(s^2).$$

The θ_2 and θ_3 numerators are closely related, and both are of the form

$$(23s\varepsilon_m\varepsilon_l^2 + 3(36 + \sqrt{31})\lambda_1)(23s\varepsilon_m\varepsilon_l^2 + 3(10 + \sqrt{31})\lambda_1)p(s^2).$$

The denominator has s^2 and the cancelling factor into a fifth-order polynomial.

Using the other valid option for λ_1:

$$\lambda_1 = -\sqrt{\frac{g\varepsilon_m^2\varepsilon_l^3(987 + 78\sqrt{31})}{297}}$$

and rearranging and substituting for g, as before, gives the cancelling pole as

$$23s\varepsilon_m\varepsilon_l^2 + (30 - 3\sqrt{31})\lambda_1.$$

The form of all the numerators and the denominator is very similar to the form for the first option, but with some sign changes which reflect the sign change within the expression for λ_1, as can be seen from the results shown in Fig. 4.

```
The 1 numerator is:
2/247960114075*(4183622950*el^10*em^5*s^5-
12215059650*el^8*r1*em^4*s^4-545689950*
el^8*r1*em^4*s^4*31^(1/2)-4017117555*
el^6*r1^2*em^3*s^3+554085180*el^6*r1^2*
em^3*s^3*31^(1/2)+18130411710*el^4*r1^3*
em^2*s^2+997024815*el^4*r1^3*em^2*s^2*
31^(1/2)-722082033*r1^4*el^2*em*s-
95616612*r1^4*el^2*em*s*31^(1/2)-
3004022376*r1^5-354797739*r1^5*31^(1/2))*
(30*r1+3*r1*31^(1/2)+23*s*em*el^2)/el^12/em^7

The 2 numerator is:
-6/93746735*(29095*s^2*em^2*el^4-
35829*r1^2+1944*r1^2*31^(1/2))*
(30*r1+3*r1*31^(1/2)+23*s*em*el^2)*
(-30*r1-3*r1*31^(1/2)+23*s*em*el^2)*
s^2/el^9/em^5

The 3 numerator is:
18/93746735*(2645*s^2*em^2*el^4-
2961*r1^2-234*r1^2*31^(1/2))*
(108*r1-3*r1*31^(1/2)+23*s*em*el^2)*
(30*r1+3*r1*31^(1/2)+23*s*em*el^2)*
s^2/el^9/em^5

The 4 numerator is:
-6/93746735*(2645*s^2*em^2*el^4+
8883*r1^2+702*r1^2*31^(1/2))*
(30*r1+3*r1*31^(1/2)+23*s*em*el^2)*
(108*r1-3*r1*31^(1/2)+23*s*em*el^2)*
s^2/el^9/em^5

The denominator is:
1/247960114075*(10780874525*el^10*
em^5*s^5-44956456650*el^8*r1*em^4*s^4-
1406201025*el^8*r1*em^4*s^4*31^(1/2)-
5926849875*el^6*r1^2*em^3*s^3+2146258800*
el^6*r1^2*em^3*s^3*31^(1/2)+81192713310*
el^4*r1^3*em^2*s^2+4464937215*el^4*r1^3*
em^2*s^2*31^(1/2)-5776656264*r1^4*el^2*em*s-
764932896*r1^4*el^2*em*s*31^(1/2)-24032179008*
r1^5-2838381912*r1^5*31^(1/2))*
(30*r1+3*r1*31^(1/2)+23*s*em*el^2)*s^2/em^6/el^12
```

Fig. 3. Symbolic results for $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 = 0$—case 1.

6. Numerical results ($\lambda_i = 0$)

We will now consider the cases when two out of the three frictions are nonzero, so that only one of the frictions is set to zero [5]. This clearly leads to a more complex form for the controllability condition in Eq. (4).

6.1. $\lambda_1 = 0$ $\lambda_2 \neq 0$ $\lambda_3 \neq 0$

In this case the determinant of C gives rise to the condition for uncontrollability:

$$19440\lambda_3^2 + 23328\lambda_2\lambda_3 - 18711\lambda_2^2 + 2116g\varepsilon_m^2\varepsilon_l^3 = 0.$$

```
The first numerator
2/247960114075*(4183622950*el^10*s^5*em^5-
12215059650*lam1*el^8*em^4*s^4+
545689950*lam1*el^8*em^4*s^4*31^(1/2)-
4017117555*lam1^2*el^6*em^3*s^3-
554085180*lam1^2*el^6*em^3*s^3*31^(1/2)+
18130411710*lam1^3*el^4*em^2*s^2-
997024815*lam1^3*el^4*em^2*s^2*31^(1/2)-
722082033*lam1^4*el^2*em*s+
95616612*lam1^4*el^2*em*s*31^(1/2)-
3004022376*lam1^5+354797739*lam1^5*31^(1/2))*
(30*lam1-3*lam1*31^(1/2)+23*s*el^2*em)/
em^7/el^12


The second numerator
-6/93746735*(29095*el^4*em^2*s^2-35829*lam1^2-
1944*31^(1/2)*lam1^2)*
(30*lam1-3*lam1*31^(1/2)+23*s*el^2*em)*
(-30*lam1+3*lam1*31^(1/2)+23*s*el^2*em)
*s^2/el^9/em^5


The third numerator
18/93746735*(2645*el^4*em^2*s^2-2961*lam1^2+
234*31^(1/2)*lam1^2)*
(30*lam1-3*lam1*31^(1/2)+23*s*el^2*em)*
(108*lam1+3*lam1*31^(1/2)+23*s*el^2*em)*
s^2/el^9/em^5


The fourth numerator
-6/93746735*(2645*el^4*em^2*s^2+8883*lam1^2-
702*31^(1/2)*lam1^2)*
(108*lam1+3*lam1*31^(1/2)+23*s*el^2*em)*
(30*lam1-3*lam1*31^(1/2)+23*s*el^2*em)*
s^2/el^9/em^5


The denominator
1/247960114075*(10780874525*el^10*s^5*em^5-
44956456650*lam1*el^8*em^4*s^4+
1406201025*lam1*el^8*em^4*s^4*31^(1/2)-
5926849875*lam1^2*el^6*em^3*s^3-
2146258800*lam1^2*el^6*em^3*s^3*31^(1/2)+
81192713310*lam1^3*el^4*em^2*s^2-
4464937215*lam1^3*el^4*em^2*s^2*31^(1/2)-
5776656264*lam1^4*el^2*em*s+
764932896*lam1^4*el^2*em*s*31^(1/2)-
24032179008*lam1^5+2838381912*lam1^5*31^(1/2))*
(30*lam1-3*lam1*31^(1/2)+23*s*el^2*em)*
s^2/em^6/el^12
```

Fig. 4. Symbolic results for $\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0$—case 2.

Clearly there are two roots for λ_2 and λ_3 for uncontrollability. Solving for λ_3 and selecting the negative root,

$$\lambda_3 = -\tfrac{3}{5}\lambda_2 - \tfrac{23}{540}\sqrt{729\lambda_2^2 - 60g\varepsilon_m^2\varepsilon_l^3}. \tag{15}$$

Obtaining the transfer functions symbolically gives the set of ratios of polynomials, which must contain a common factor. Again, difficulties are caused by the fractional powers which occur throughout the transfer functions, and the software is therefore unable to extract the cancelling pole.

Thus, a numerical investigation of the cancelling pole has been undertaken to enable the form of the cancelling pole to be deduced. For simplicity, at this stage, g, the acceleration due to gravity, is taken to be 10 $\mathrm{m/s^2}$:

ε_l	ε_m	λ_2	λ_3	Cancelling pole
1.00	1.00	−1.10	−0.055	$s-\frac{297}{20}+\frac{\sqrt{28209}}{20}$
1.00	0.10	−1.00	−0.545	$s-135+5\sqrt{723}$
0.10	1.00	−0.07	−0.029	$s-\frac{8991}{100}+\frac{\sqrt{65838081}}{100}$
0.25	0.75	−0.13	−0.030	$s-36+2\sqrt{87}\sqrt{2}$

Using the numerical results, it is possible to deduce a form for the cancelling pole, as follows:

$$s+\frac{27}{2\varepsilon_m\varepsilon_l^2}\left(\lambda_2+\sqrt{729\lambda_2^2-60g\varepsilon_m^2\varepsilon_l^3}\right).$$

Substituting for $\sqrt{729\lambda_2^2-60g\varepsilon_m^2\varepsilon_l^3}$ from Eq. (15) gives the cancelling pole in the form

$$s+\frac{27}{46\varepsilon_m\varepsilon_l^2}(11\lambda_2-20\lambda_3). \tag{16}$$

This result is supported by results reported in the previous section. For example, if $\lambda_3=0$, the cancelling pole correctly reduces to

$$s+\frac{297\lambda_2}{46\varepsilon_m\varepsilon_l^2},$$

which was the form obtained in Eq. (13).

6.2. $\lambda_1 \neq 0$ $\lambda_2 = 0$ $\lambda_3 \neq 0$

In this case the determinant of C gives rise to the condition for uncontrollability:

$$\begin{aligned}&297\lambda_1^4-1116\lambda_1^3\lambda_3-5760\lambda_1^2\lambda_3^2-1974g\varepsilon_m^2\varepsilon_l^3\lambda_1^2+4380g\varepsilon_m^2\varepsilon_l^3\lambda_1\lambda_3\\&\quad+24300g\varepsilon_m^2\varepsilon_l^3\lambda_3^2+2645g^2e_l^6e_m^4=0.\end{aligned} \tag{17}$$

This is clearly a much more difficult equation to solve, so first consider the case when $\lambda_1=\lambda_3=\lambda$. This gives the simplified condition for uncontrollability:

$$6579\lambda^4-26706g\varepsilon_m^2\varepsilon_l^3\lambda^2-2645g^2e_l^6e_m^4=0.$$

The solution of this equation for λ, recalling that friction has been defined to be negative, is

$$\lambda=-\frac{\varepsilon_m\varepsilon_l}{2193}\sqrt{(9761043+1017552\sqrt{101})g\varepsilon_l}. \tag{18}$$

Results from the numerical investigation are shown in the following table:

ε_l	ε_m	λ	Cancelling pole
1.00	1.00	−1.0	$23s + 105 - 12\sqrt{101}$
1.00	0.10	−0.5	$23s + 525 - 60\sqrt{101}$
0.10	1.00	−0.2039	$2300s + 214095 - 24468\sqrt{101}$
0.10	0.10	−0.5	$23s + 525000 - 6000\sqrt{101}$

It has been deduced that the cancelling pole may be written as

$$s - \frac{105}{23\varepsilon_m\varepsilon_l^2}\lambda + \frac{12\sqrt{101}}{23\varepsilon_m\varepsilon_l^2}\lambda. \tag{19}$$

Now consider the case when $\lambda_1 \neq \lambda_3$. Solving the full condition for uncontrollability for λ_3 and taking the only valid negative root gives

$$\lambda_3 = -\frac{\lambda_1(365g\varepsilon_m^2\varepsilon_l^3 - 93\lambda_1^2)}{30(135g\varepsilon_m^2\varepsilon_l^3 - 32\lambda_1^2)} - \frac{\sqrt{9\lambda_1^2 - 15g\varepsilon_m^2\varepsilon_l^3}(345g\varepsilon_m^2\varepsilon_l^3 - 79\lambda_1^2)}{30(135g\varepsilon_m^2\varepsilon_l^3 - 32\lambda_1^2)}.$$

Results from the numerical investigation are shown in the following table:

ε_l	ε_m	λ_1	λ_3	Cancelling pole
0.50	0.50	−1.0	−0.1269	$5750s - 25737 + 3\sqrt{23}\sqrt{1351967}$
0.50	0.50	−0.8	−0.024	$575s - 4152 + 24\sqrt{7849}$
1.00	1.00	−5.0	−0.3727	$46000s - 199371 + 3\sqrt{3}\sqrt{73}\sqrt{4316771}$
1.00	1.00	−4.5	−0.1214	$23000s - 118611 + 9\sqrt{46358041}$

It has been deduced that the cancelling pole may be written as

$$s + \frac{3}{23\varepsilon_m\varepsilon_l^2}\left[5(2\lambda_1 - 9\lambda_3) + \sqrt{31\lambda_1^2 - 440\lambda_1\lambda_3 + 2025\lambda_3^2}\right]. \tag{20}$$

It is probable that the square root term will cause difficulties for the symbolic manipulation package. Clearly, if $\lambda_1 = \lambda_3$ then Eq. (20) reduces to the form in Eq. (19).

6.3. $\lambda_1 \neq 0$ $\lambda_2 \neq 0$ $\lambda_3 = 0$

In this case the determinant of C gives rise to the condition for uncontrollability:

$$\begin{aligned} &g\varepsilon_m^2\varepsilon_l^3(1188\lambda_1^4 + 81648\lambda_1\lambda_2^3) - 891\lambda_1^4\lambda_2^2 - 3780\lambda_1^3\lambda_2^3 + g\varepsilon_m^2\varepsilon_l^3(11376\lambda_1^3\lambda_2 + 48906\lambda_1^2\lambda_2^2) \\ &\quad - 44688g^2\varepsilon_m^4\varepsilon_l^6\lambda_1\lambda_2 - 93555g^2\varepsilon_m^4\varepsilon_l^6\lambda_2^2 - 7896g^2\varepsilon_m^4\varepsilon_l^6\lambda_1^2 + 10580g^3\varepsilon_m^6\varepsilon_l^9 = 0. \end{aligned} \tag{21}$$

Again, this is a complicated expression, and it is apparently not possible to simplify by considering the case when $\lambda_1 = \lambda_2 = \lambda$ since this gives rise to complex values for λ. This is due to the manner in which the software solves a cubic polynomial with symbolic coefficients, as very small imaginary parts are introduced during the process. However, if λ_1 and λ_2 are not equal, and we solve for λ_1, very small imaginary parts are again introduced by the manipulation software. As an illustration, the case when $\lambda_1 = \lambda_2 = \lambda$ gives the following solution for Eq. (21):

$$\lambda = \left(-\frac{3568720826617579}{4835703278458516698824704} + \frac{6113736769972699}{3094850098213450687247781056i}\right)$$
$$\times e_m e_l \sqrt{((542036790667217223\,68 + 2903917271836301824i)ge_l)}$$
$$= (-0.738\mathrm{e}^{-9} + 0.198\mathrm{e}^{-10}\mathrm{i})e_m e_l \sqrt{((0.542\mathrm{e}^{20} + 0.290\mathrm{e}^{19}\mathrm{i})ge_l)}.$$

Results from the numerical investigation for the case when $\lambda_1 \neq \lambda_2$ are shown in the following table:

ε_l	ε_m	λ_1	λ_2	Cancelling pole
1.00	1.00	−0.5	−0.9655	$s - 6.991$
1.00	0.50	−0.5	−0.4304	$s - 7.135$
0.50	1.00	−1.0	−0.1482	$s - 11.13$
0.50	0.50	−1.0	−0.1683	$s - 9.912$

In this case, it is not possible to deduce a rational numerical form for the cancelling pole, which means that a full symbolic form for the cancelling pole cannot be suggested, however it is postulated that the cancelling pole is of the form

$$s + \frac{30\lambda_1}{23\varepsilon_m\varepsilon_l^2} + \frac{297\lambda_2}{46\varepsilon_m\varepsilon_l^2} + \mathrm{II}. \tag{22}$$

The deduction of the form of the cancelling pole is supported by results from earlier work [4], which have been reported in the previous section. In the case when only λ_1 is nonzero, Eq. (22) reduces to

$$s + \frac{30\lambda_1}{23\varepsilon_m\varepsilon_l^2} + \mathrm{II}.$$

In the case when only λ_1 is nonzero, Eq. (22) reduces to

$$s + \frac{297\lambda_2}{46\varepsilon_m\varepsilon_l^2} + \mathrm{II}.$$

From Eq. (13) it is known that the symbolic form of the cancelling pole when only λ_1 is nonzero is

$$s + \frac{(30 + 3\sqrt{31})\lambda_1}{23\varepsilon_m\varepsilon_l^2}$$

and for the case when only λ_2 is nonzero the symbolic form of the cancelling pole is

$$s + \frac{297\lambda_2}{46\varepsilon_m\varepsilon_l^2}.$$

Thus, we can deduce a general form for the second part of the cancelling pole for the case when both λ_1 and λ_2 are nonzero:

$$\mathrm{II} = \lambda_1 \frac{3\sqrt{31}}{23\varepsilon_m\varepsilon_l^2} + \lambda_1\lambda_2 f(\lambda_1, \lambda_2, \varepsilon_m, \varepsilon_l).$$

The second part of the cancelling pole clearly has to contain a term in $\lambda_1\lambda_2$ to satisfy earlier known results, and it is possible that it is this term which is causing the symbolic manipulation package some difficulties.

7. Symbolic results

Having obtained, numerically, the form of the cancelling pole for the three cases, it is now appropriate to revisit the symbolic forms of the transfer functions. Since it is apparently the fractional powers which cause the manipulation package problems, the uncontrollability conditions need to be rearranged to avoid their use, where possible.

7.1. $\lambda_1 = 0$ $\lambda_2 \neq 0$ $\lambda_3 \neq 0$

In this case, it is clear from Eq. (15) that the solution of the uncontrollability condition for λ_2 contains fractional powers. The solution for λ_3 also contains fractional powers, so in either case the symbolic manipulation package is unable to extract the cancelling pole. Instead, the condition has been solved for g, the acceleration due to gravity, which gives an expression which does not contain fractional powers:

$$g = -\frac{243(4\lambda_3 + 7\lambda_2)(20\lambda_3 - 11\lambda_2)}{2116\varepsilon_m^2\varepsilon_l^3}.$$

Substituting for g in the numerators and denominator of the transfer functions, and factorising, gives the results shown in Fig. 5. The common cancelling pole is

$$540\lambda_3 - 297\lambda_2 - 46s/\varepsilon_m\varepsilon_l^2.$$

Rearranging Eq. (16) gives

$$-46\varepsilon_m\varepsilon_l^2 s - 297\lambda_2 + 540\lambda_3,$$

so the form of the cancelling pole deduced from the numerical results is correct.

7.2. $\lambda_1 \neq 0$ $\lambda_2 = 0$ $\lambda_3 \neq 0$

When $\lambda_1 = \lambda_3 = \lambda$ the uncontrollability condition in Eq. (18) in terms of λ clearly contains fractional powers. Rearranging so that g is the subject, we get an expression with only one fractional power:

$$g = \frac{3(-4451 + 464\sqrt{101})\lambda^2}{2645\varepsilon_l^3\varepsilon_m^2}.$$

Substituting for g in the numerators and denominator of the transfer functions, and factorising, gives the results shown in Fig. 6. The common cancelling pole is

$$23s\varepsilon_m\varepsilon_l^2 - 105\lambda + 12\sqrt{101}\lambda.$$

Rearranging Eq. (19) gives

$$23\varepsilon_m\varepsilon_l^2 s - 105\lambda + 12\sqrt{101}\lambda,$$

so the form of the cancelling pole deduced from the numerical results is correct.

```
The 1 numerator with the non-controllability condition is:
1/317388946016*(540*r3-297*r2-46*s*em*el^2)*(5509980288000*
r3^5+22866418195200*r3^4*r2-1129961664000*el^2*s*em*r3^4+
1396452337920*r3^3*el^4*em^2*s^2-3151723841280*el^2*s*em*r2*
r3^3+20469576769920*r3^3*r2^2-749425343616*el^2*s*em*r2^2*r3^2-
17404650234720*r3^2*r2^3+4238743742784*r3^2*el^4*em^2*s^2*r2-
275632191360*r3^2*s^3*em^3*el^6-426504548736*r3*s^3*r2*em^3*el^6+
35998746240*r3*em^4*el^8*s^4+1731515749488*r3*el^4*em^2*s^2*r2^2-
17169615138060*r3*r2^4+2109920917104*el^2*s*em*r2^3*r3+8932840308315*
r2^5+97740625752*s^3*em^3*el^6*r2^2-305988729354*el^2*s*em*r2^4-
5355037376*em^5*el^10*s^5+88949140896*em^4*el^8*s^4*r2-2466888402132*
el^4*em^2*s^2*r2^3)/em^7/el^12

The 2 numerator with the non-controllability condition is:
-3/149994776*s^2*(540*r3-297*r2-46*s*em*el^2)*(94478400*r3^3+
278711280*r3^2*r2-19375200*el^2*s*em*r3^2+6119472*r3*el^4*em^2*
s^2-30791664*el^2*r2*s*em*r3+107469180*r3*r2^2+5451138*s*em*
el^2*r2^2-1070696*s^3*em^3*el^6+12753132*s^2*r2*em^2*el^4-
159137055*r2^3)/em^5/el^9

The 3 numerator with the non-controllability condition is:
-9/149994776*s^2*(540*r3-297*r2-46*s*em*el^2)*(31492800*r3^3+
92903760*r3^2*r2-6458400*el^2*s*em*r3^2+363952*r3*el^4*em^2*s^2-
10263888*el^2*r2*s*em*r3+35823060*r3*r2^2+97336*s^3*em^3*el^6+1817046*
s*em*el^2*r2^2+1318268*s^2*r2*em^2*el^4-53045685*r2^3)/em^5/el^9

The 4 numerator with the non-controllability condition is:
-3/149994776*s^2*(540*r3-297*r2-46*s*em*el^2)*(94478400*r3^3+
278711280*r3^2*r2-19375200*el^2*s*em*r3^2-3478704*r3*el^4*em^2*
s^2+107469180*r3*r2^2-30791664*el^2*r2*s*em*r3+5451138*s*em*el^2*r2^2-
159137055*r2^3-4043676*s^2*r2*em^2*el^4-97336*s^3*em^3*el^6)/em^5/el^9

The denominator with the non-controllability condition is:
1/158694473008*s^2*(540*r3-297*r2-46*s*em*el^2)*(11019960576000*r3^5+
45732836390400*r3^4*r2-2259923328000*el^2*s*em*r3^4-6303447682560*
el^2*s*em*r2*r3^3+40939153539840*r3^3*r2^2+1443469688640*r3^3*el^4*em^2
*s^2-1498850687232*el^2*s*em*r2^2*r3^2-274528401120*r3^2*s^3*em^3*
el^6+4496654273328*r3^2*el^4*em^2*s^2*r2-34809300469440*r3^2*r2^3+
4219841834208*el^2*s*em*r2^3*r3+1928049201036*r3*el^4*em^2*s^2*r2^2+
23761858992*r3*em^4*el^8*s^4-34339230276120*r3*r2^4-413211760560*
r3*s^3*r2*em^3*el^6+17865680616630*r2^5+64290671340*em^4*el^8*s^4*r2-
611977458708*el^2*s*em*r2^4+117622647450*s^3*em^3*el^6*r2^2-3449879848*
em^5*el^10*s^5-2660822247699*el^4*em^2*s^2*r2^3)/em^6/el^12
```

Fig. 5. Symbolic results for $\lambda_1 = 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0$.

For the general case, when $\lambda_1 \neq \lambda_3$, solving Eq. (17) for g does not eliminate the fractional powers:

$$g = \frac{3}{2645\varepsilon_m^2\varepsilon_l^3}[329\lambda_1^2 - 730\lambda_1\lambda_3 - 4050\lambda_3^2 + 2\sqrt{(31\lambda_1^2 - 440\lambda_1\lambda_3 + 2025\lambda_3^2)(45\lambda_3 + 13\lambda_1)^2}].$$

It is therefore not possible for the symbolic manipulation package to extract the cancelling pole symbolically for the general case when $\lambda_1 \neq \lambda_3$.

7.3. $\lambda_1 \neq 0\ \ \lambda_2 \neq 0\ \ \lambda_3 = 0$

As explained earlier, the solution of Eq. (21) for $\lambda_1,\ \lambda_2$ or for λ, where $\lambda_1 = \lambda_2 = \lambda$, gives solutions which contain very small imaginary parts. Substituting either the actual solution, or the solution with

```
The 1 numerator with the non-controllability condition is:
2/247960114075*(4183622950*el^10*em^5*s^5-64895127900*el^8*r1*
em^4*s^4-2182759800*el^8*r1*em^4*s^4*101^(1/2)+211194603495*
el^6*r1^2*em^3*s^3-7354221480*el^6*r1^2*em^3*s^3*101^(1/2)-
841605564555*el^4*r1^3*em^2*s^2+84880155420*el^4*r1^3*em^2*s^2
*101^(1/2)+706969654587*r1^4*el^2*em*s-71847295488*r1^4*el^2*
em*s*101^(1/2)-3670364366001*r1^5+365081709924*r1^5*101^(1/2))*
(23*el^2*em*s-105*r1+12*r1*101^(1/2))/em^7/el^12

The 2 numerator with the non-controllability condition is:
-6/93746735*(669185*s^3*em^3*el^6-8625345*r1*el^4*em^2*s^2-
349140*r1*el^4*em^2*s^2*101^(1/2)+10347723*el^2*r1^2*em*s-
359352*el^2*r1^2*em*s*101^(1/2)-44386947*r1^3+4486428*r1^3*
101^(1/2))*(23*el^2*em*s-105*r1+12*r1*101^(1/2))*s^2/el^9/em^5

The 3 numerator with the non-controllability condition is:
18/93746735*(-21160*s*em*el^2*r1+40059*r1^2-4176*r1^2*101^(1/2)+
2645*el^4*em^2*s^2)*(23*el^2*em*s+243*r1-12*r1*101^(1/2))*
(23*el^2*em*s-105*r1+12*r1*101^(1/2))*s^2/el^9/em^5

The 4 numerator with the non-controllability condition is:
-6/93746735*(63480*s*em*el^2*r1-120177*r1^2+12528*r1^2*101^(1/2)+
2645*el^4*em^2*s^2)*(23*el^2*em*s-105*r1+12*r1*101^(1/2))*
(23*el^2*em*s+243*r1-12*r1*101^(1/2))*s^2/el^9/em^5

The denominator with the non-controllability condition is:
1/247960114075*(10780874525*el^10*em^5*s^5-182491312125*el^8*
r1*em^4*s^4-5624804100*el^8*r1*em^4*s^4*101^(1/2)+877132717875*
el^6*r1^2*em^3*s^3-29543909400*el^6*r1^2*em^3*s^3*101^(1/2)-
3562950696435*el^4*r1^3*em^2*s^2+358642987740*el^4*r1^3*em^2*
s^2*101^(1/2)+5655757236696*r1^4*el^2*em*s-574778363904*r1^4*
el^2*em*s*101^(1/2)-29362914928008*r1^5+2920653679392*r1^5*
101^(1/2))*(23*el^2*em*s-105*r1+12*r1*101^(1/2))*s^2/el^12/em^6
```

Fig. 6. Symbolic results for $\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 \neq 0$.

```
Pls input the value of arm friction: r1=-.5
Pls input the value of arm friction: r2=-.9655

Numerators:
[ .7761*(s+16.72)*(s+5.679)*(s+2.224)*(s-2.180)*(s-4.840)*(s-6.991)]
[ -.9851*(s+11.55)*(s+4.268)*s^2*(s-3.561)*(s-6.991)]
[ .2687*(s+5.285)*s^2*(s-6.992)*(s^2-20.60*s+121.8)]
[ -.8955e-1*s^2*(s-6.991)*(s-44.22)*(s^2+1.863*s+43.67)]

Denominator:
1.000*(s+19.32)*(s+6.485)*(s+3.158)*s^2*(s-3.046)*(s-5.738)*(s-6.991)]
```

Fig. 7. Numerical results for $\varepsilon_m = \varepsilon_l = 1,\ \lambda_1 = -0.5,\ \lambda_2 = -0.9655$.

the imaginary part omitted, does not allow the cancelling pole and zeros to be factorised out by the symbolic software.

Obtaining the transfer functions symbolically, substituting in the uncontrollability condition in terms of g, and then substituting in values for ε_m, ε_l, λ_1 and λ_2 prior to factorising, allows the results from the numerical investigation to be confirmed. By way of illustration, the results for this process for the case when $\varepsilon_m = \varepsilon_l = 1,\ \lambda_1 = -0.5$ and $\lambda_2 = -0.9655$ are as depicted in Fig. 7. The

```
The denominator with the non-controllability condition is:
-.9783e-51*(.9e31*i*s^5+.1312e34*r^5-.8e31*s^5-.141e33*
r^3*s^2-.10e33*r^2*s^3-.17e32*r^4*s-.43e33*i*r^2*s^3+
.17e31*i*r^3*s^2-.27e31*i*r^4*s+.22e33*i*r*s^4)*
(.7820e23*r+.9715e22*s-.2e19*i*s)*s^2

The 1 numerator with the non-controllability condition is:
-.1957e-50*(.5e32*r*s^4+.1640e33*r^5+.6e31*s^5-.3104e32*
r^3*s^2-.6e31*i*s^5+.55e32*r^2*s^3-.21e31*r^4*s+.16e30*i*
r^3*s^2-.15e32*i*r^2*s^3-.37e30*i*r^4*s-.10e33*i*r*s^4)*
(.7820e23*r+.9715e22*s-.2e19*i*s)

The 2 numerator with the non-controllability condition is:
(-.2387e-48+.4282e-53*i)*(.30e27*s^3+.1053e30*r^3+.12e28*
i*s^2*r+.35e27*i*s*r^2+.9e26*i*s^3-.446e28*s*r^2-.61e28*s^2*r)*
(.7820e23*r+.9715e22*s-.2e19*i*s)

The 3 numerator with the non-controllability condition is:
(-.7163e-48+.1285e-52*i)*(-.33e26*s^3+.3511e29*r^3-.10e27*
i*s^2*r-.9e26*i*s*r^2+.5e25*i*s^3-.34e27*s*r^2-.103e28*s^2*r)*
( .7820e23*r+.9715e22*s-.2e19*i*s)*s^2

The 4 numerator with the non-controllability condition is:
(-.2387e-48+.4282e-53*i)*(.33e26*s^3+.1053e30*r^3+.3e27*i*s^2*
r+.31e27*i*s*r^2-.5e25*i*s^3-.120e28*s*r^2+.16e28*s ^2*r)*
(.7820e23*r+. 9715e22*s-.2e19*i*s )*s^2
```

Fig. 8. Results for $\lambda_1 = \lambda_2 = \lambda$, $\varepsilon_m = \varepsilon_l = 1$.

common cancelling pole is

$$s - 6.991,$$

which is the value previously obtained during the numerical investigation. Substituting the same numerical values into Eq. (22) gives

$$s - 6.8859 + \mathrm{II}.$$

Hence, we can deduce that the value of the second part of the cancelling pole is two orders of magnitude less than that of the first part, and in this case,

$$\mathrm{II} = -0.1051.$$

Substituting the uncontrollability condition in terms of g, for the case when $\lambda_1 = \lambda_2 = \lambda$, and then substituting $\varepsilon_m = \varepsilon_l = 1$, and converting to variable precision arithmetic, we obtain the factorised denominator and numerators for the transfer functions shown in Fig. 8. The common cancelling pole is

$$0.7820\mathrm{e}^{23}\lambda + 0.9715\mathrm{e}^{22}s - 0.2\mathrm{e}^{19}\mathrm{i}s,$$

which rearranges to

$$s(1 - 2.0587\mathrm{e}^{-4}\mathrm{i}) + 8.0494\lambda.$$

Substituting the same numerical values into Eq. (22) gives

$$s + 7.7609\lambda + \mathrm{II}.$$

In this case, the second part of the cancelling pole has the form

$$\mathrm{II} = 0.2885\lambda - 2.0587\mathrm{e}^{-4}\mathrm{i}s.$$

In both cases, II has a negative real part of similar magnitude, recalling that friction has been defined to be negative. For the second case, II has a complex part which is three orders of magnitude smaller than the real part, and can perhaps be attributed to a quirk of the symbolic manipulation package.

8. Conclusions

A symbolic-numeric investigation of the cancelling pole and zeros of the triple-inverted pendulum has been undertaken for the cases when one of the three frictions λ_1, λ_2 and λ_3 are nonzero, and also for the cases when two out of the three frictions are nonzero.

For the case when only λ_3 is nonzero, the system is always controllable and there is therefore no pole-zero cancellation. For the case when only λ_2 is nonzero, the expected cancelling pole has been extracted algebraically from each of the numerators and the denominator, and is identical to the form deduced from the numerical investigation. The innovative step of substituting for g, which is a known constant, has enabled the pole to be extracted symbolically, and was suggested by the results from the numerical investigation. For the case when only λ_1 is nonzero, the expected cancelling pole has again been extracted algebraically using this strategy, and the results support those obtained numerically.

For the case when $\lambda_1 = 0$ the expected cancelling pole has been extracted algebraically from each numerator and the denominator using the strategy employed for the previous cases, and is identical to the form deduced from the numerical investigation. For the case when $\lambda_2 = 0$ the expected cancelling pole has been extracted algebraically from each numerator and the denominator when $\lambda_1 = \lambda_3$. For the fully general case, it is not possible to rearrange the uncontrollability condition in such a way as to eliminate fractional powers, and hence the symbolic software is unable to extract the cancelling pole. However, the symbolic form of the cancelling pole has been deduced for the numerical investigations, and reduces to the correct form when $\lambda_1 = \lambda_3$.

When $\lambda_3 = 0$, it has not been possible to obtain the cancelling pole in rational form, leading to difficulties in deducing full algebraic form, and with symbolic manipulations. A form of the cancelling pole has been deduced, supported by results from earlier work, and particular numerical substitutions.

It is clear that the rewriting of the condition of uncontrollability to avoid the use of fractional powers has enabled the symbolic software to successfully obtain the cancelling factor for each of the five highly complex polynomials, for three of the cases investigated. Current work is concerned with further investigations of the case when $\lambda_3 = 0$, and also the case when all of the three frictions are nonzero, which is clearly an even more complex problem.

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